Logistic equation, chaos and nonlinear dynamics

Exponential growth model:

$$\frac{dN}{dt} = rN$$

N(t) – population size at time t
r – growth rate

Solution: $N(t) = N_0 e^{rt}$

Unfortunately, the solution diverges for $t \to \infty$

How to make it better?

Logistic growth (P.-F. Verhulst - 1845):

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

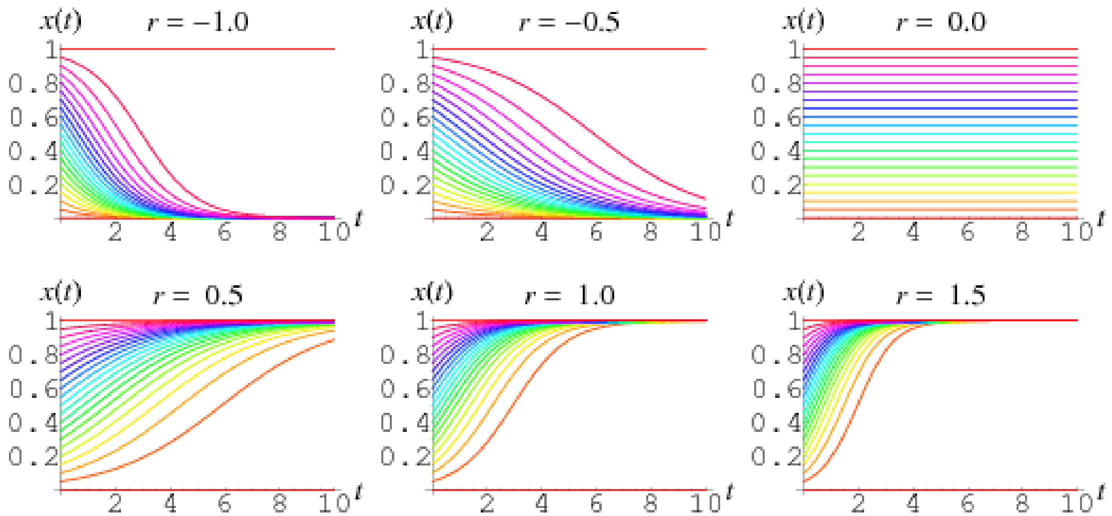
K – environmental carrying capacity

Using $x = \frac{N}{K}$ we can write

$$\frac{dx}{dt} = rx(1-x),$$

Solution:

$$x(t) = \frac{1}{1 + \left(\frac{1}{x_0} - 1\right)e^{-rt}}.$$



Discrete logistic growth

$$X_{n+1} = rX_n(1 - X_n)$$

Numerical experiment:

n	$r = 0.5$	$r = 2.0$	$r = 2.7$	$r = 3.2$	$r = 3.5$	$r = 3.8$
0	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
1	0.0450	0.1800	0.2430	0.2880	0.3150	0.3420
2	0.0215	0.2952	0.4967	0.6562	0.7552	0.8551
3	0.0105	0.4161	0.6750	0.7219	0.6470	0.4707
4	0.0052	0.4859	0.5923	0.6424	0.7993	0.9467
5	0.0026	0.4996	0.6520	0.7351	0.5614	0.1916
6	0.0013	0.5000	0.6126	0.6231	0.8618	0.5886
7	0.0006	0.5000	0.6407	0.7515	0.4168	0.9202
8	0.0003	0.5000	0.6215	0.5975	0.8508	0.2790
9	0.0002	0.5000	0.6351	0.7696	0.4443	0.7645
10	0.0001	0.5000	0.6257	0.5675	0.8641	0.6842
11	0.0000	0.5000	0.6323	0.7854	0.4109	0.8211
12	0.0000	0.5000	0.6277	0.5393	0.8472	0.5583
13	0.0000	0.5000	0.6310	0.7951	0.4531	0.9371
14	0.0000	0.5000	0.6287	0.5214	0.8673	0.2240
15	0.0000	0.5000	0.6303	0.7985	0.4029	0.6606
16	0.0000	0.5000	0.6292	0.5148	0.8420	0.8519
17	0.0000	0.5000	0.6300	0.7993	0.4657	0.4793
18	0.0000	0.5000	0.6294	0.5133	0.8709	0.9484
19	0.0000	0.5000	0.6298	0.7994	0.3936	0.1861
20	0.0000	0.5000	0.6295	0.5131	0.8353	0.5755
21	0.0000	0.5000	0.6297	0.7995	0.4814	0.9284
22	0.0000	0.5000	0.6296	0.5131	0.8738	0.2527
23	0.0000	0.5000	0.6297	0.7995	0.3860	0.7177
24	0.0000	0.5000	0.6296	0.5130	0.8295	0.7699
25	0.0000	0.5000	0.6296	0.7995	0.4950	0.6731
26	0.0000	0.5000	0.6296	0.5130	0.8749	0.8362
27	0.0000	0.5000	0.6296	0.7995	0.3830	0.5206
28	0.0000	0.5000	0.6296	0.5130	0.8271	0.9484
29	0.0000	0.5000	0.6296	0.7995	0.5005	0.1860
30	0.0000	0.5000	0.6296	0.5130	0.8750	0.5753

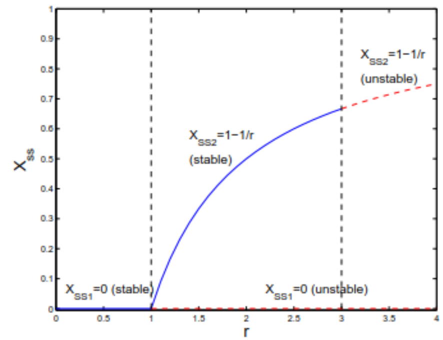
Discrete logistic growth

Find the steady state of logistic equation $X_{n+1} = X_n = X_{ss}$

One obtains $X_{ss1} = 0$ and $X_{ss2} = 1 - 1/r$

Examine stability of the steady-state solutions

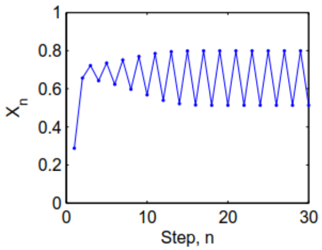
X_{ss1} is stable if $r < 1$ X_{ss2} is stable if $1 < r < 3$



Steady states as a function of r

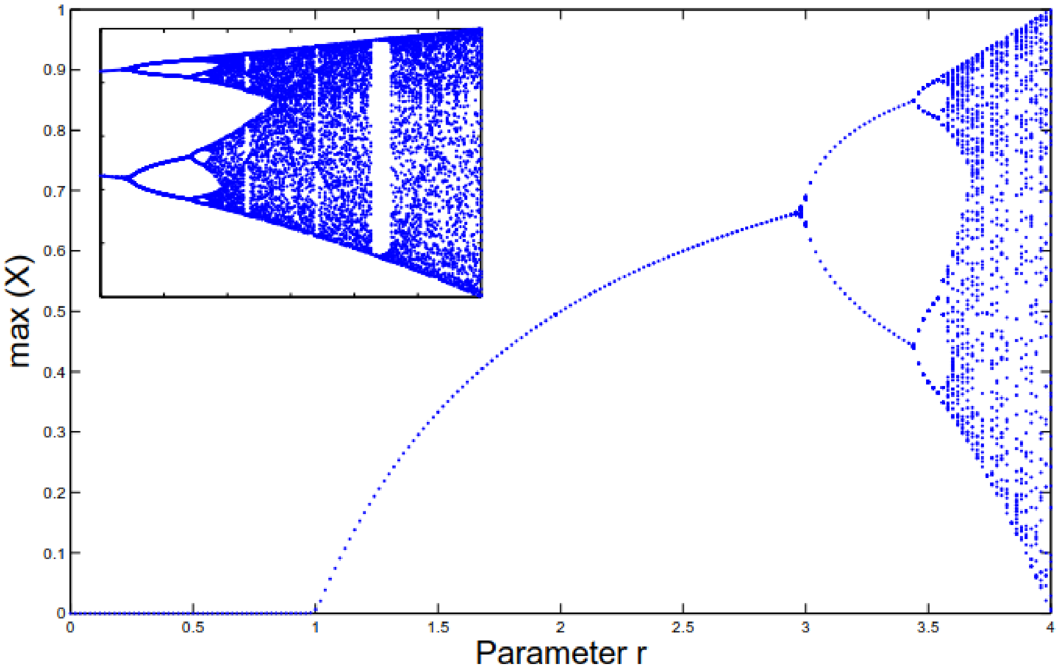
What happens for r>3 ?

Discrete logistic growth



r=3.2

(biennial oscilations)

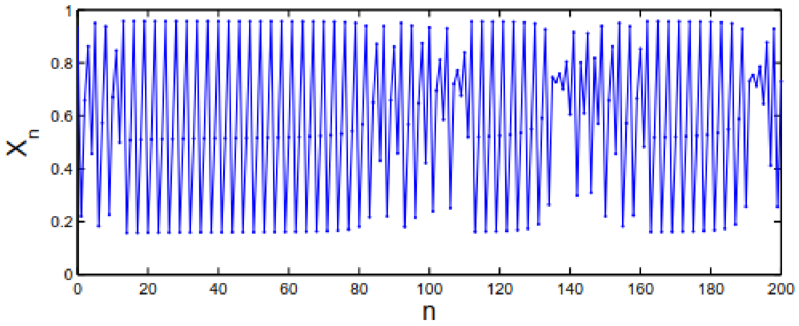


Bifurcation diagram

How to make such a diagram ?

<u>Simple equation might have a complex solution</u>

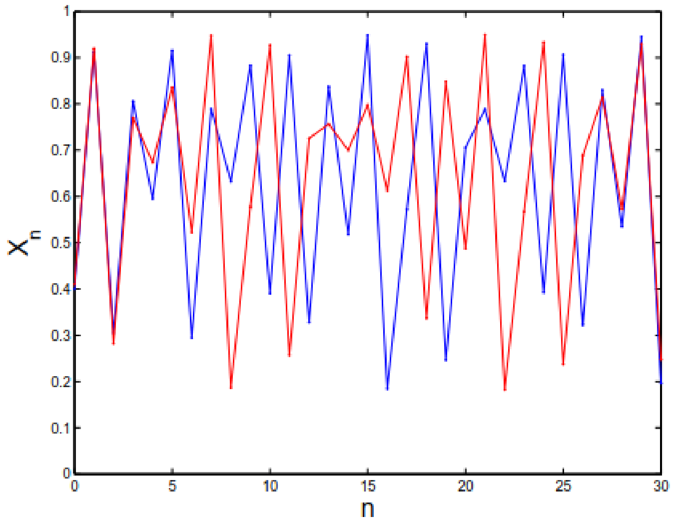
Discrete logistic growth – chaotic behaviour



r=3.8282

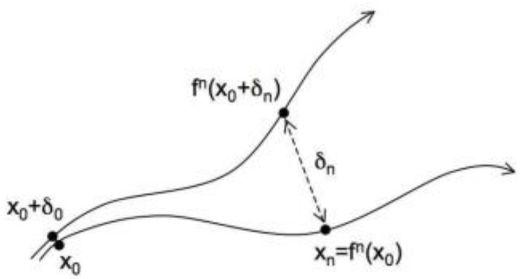
Intermittency

Sensitivity to initial conditions



$R = 3.8, x_0 = 0.4\ (blue), x_0 = 0.41\ (red)$

How to measure chaos?

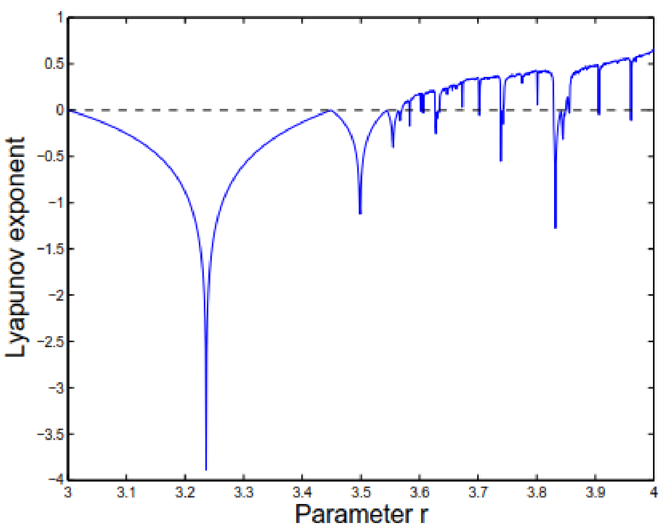


$$|\delta_n| \approx |\delta_0| \, e^{n\lambda}$$

λ – Lyapunov exponent

<u>$\lambda > 0 \rightarrow$ chaos</u>

Calculate Lyapunov exponent for the logistic equation:



$$\lambda = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|\right]$$

Feigenbaum constant

- Feigenbaum constant equals:

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669201660910299097...$$

- Feigenbaum point: $r_\infty = 3.5699456$…

Feigenbaum constant – cont.

- Feigenbaum constant appear for several dynamical systems (universality, e.g., $x_{n+1} = k\sin(x_n/\pi)$)
- Feigenbaum route to chaos was observed in some experiments or numerical calculations:
  - „Chaos in a dripping faucet”, Nunez et al.
  - Liquid hellium oscillations
  - Chua circuits
  - Populations of gypsy moth

Discrete logistic growth – extensions

Delayed logistic growth

$$X_{n+1} = rX_n(1 - X_{n-1}) \qquad \text{Meynard-Smith (1968)}$$

equivalent to:

$$\begin{aligned} X_{n+1} &= rX_n(1 - Y_n) \\ Y_{n+1} &= X_n \end{aligned}$$

Examine numerically the above set of equations.

Arms race (two nations)

$$x_{t+1} = 4ay_t(1 - y_t) \equiv f_a(y_t)$$

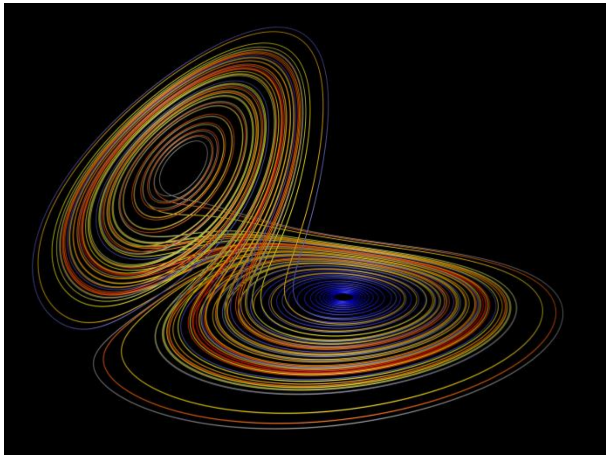
$$y_{t+1} = 4bx_t(1 - x_t) \equiv f_b(x_t)$$

A. Saperstein, "Chaos - a Model for the Outbreak of War", Nature 309, 303-305, 1984

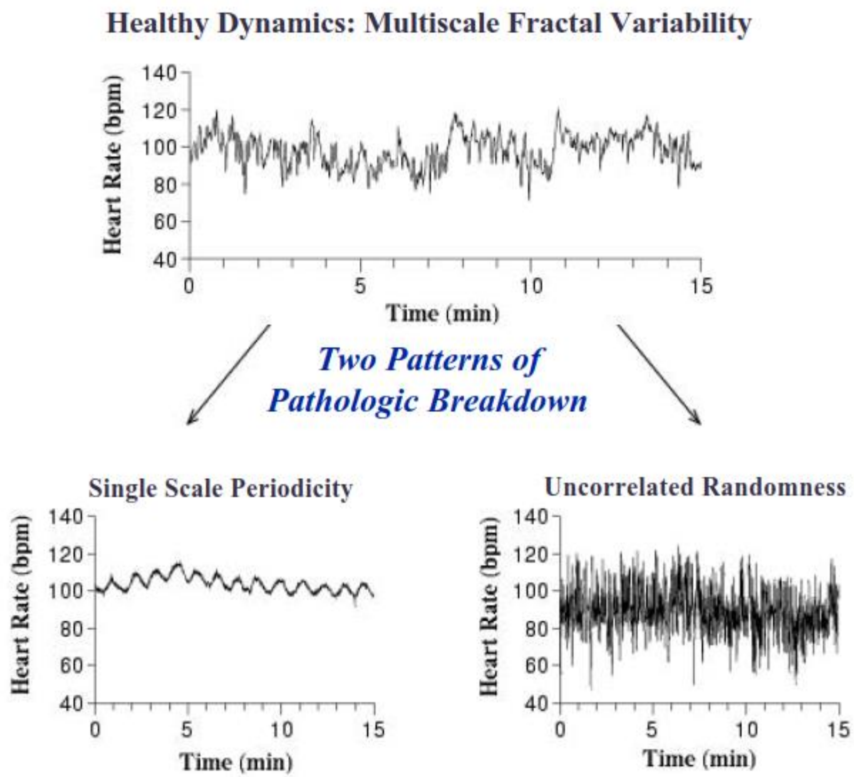
Chaos beyond logistic equation

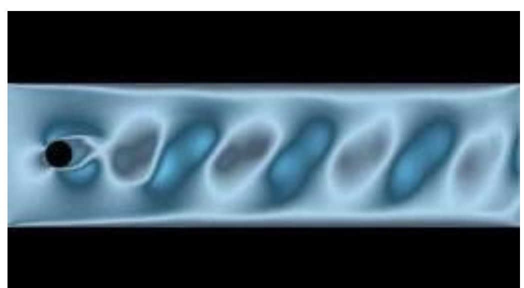
Lorenz model

$$\frac{dx}{dt} = -\sigma x + \sigma y$$
$$\frac{dy}{dt} = -y + rx - xz$$
$$\frac{dz}{dt} = -bz + xy$$



Chaos





Weather, lasers, electric cicrcuits, chemical reactions, economy, …

Summary

- Simple dynamical systems might have a very rich behaviour
- Chaos in deterministic systems
- Ubiquity of chaos – how to tame chaos ?

Literature:

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