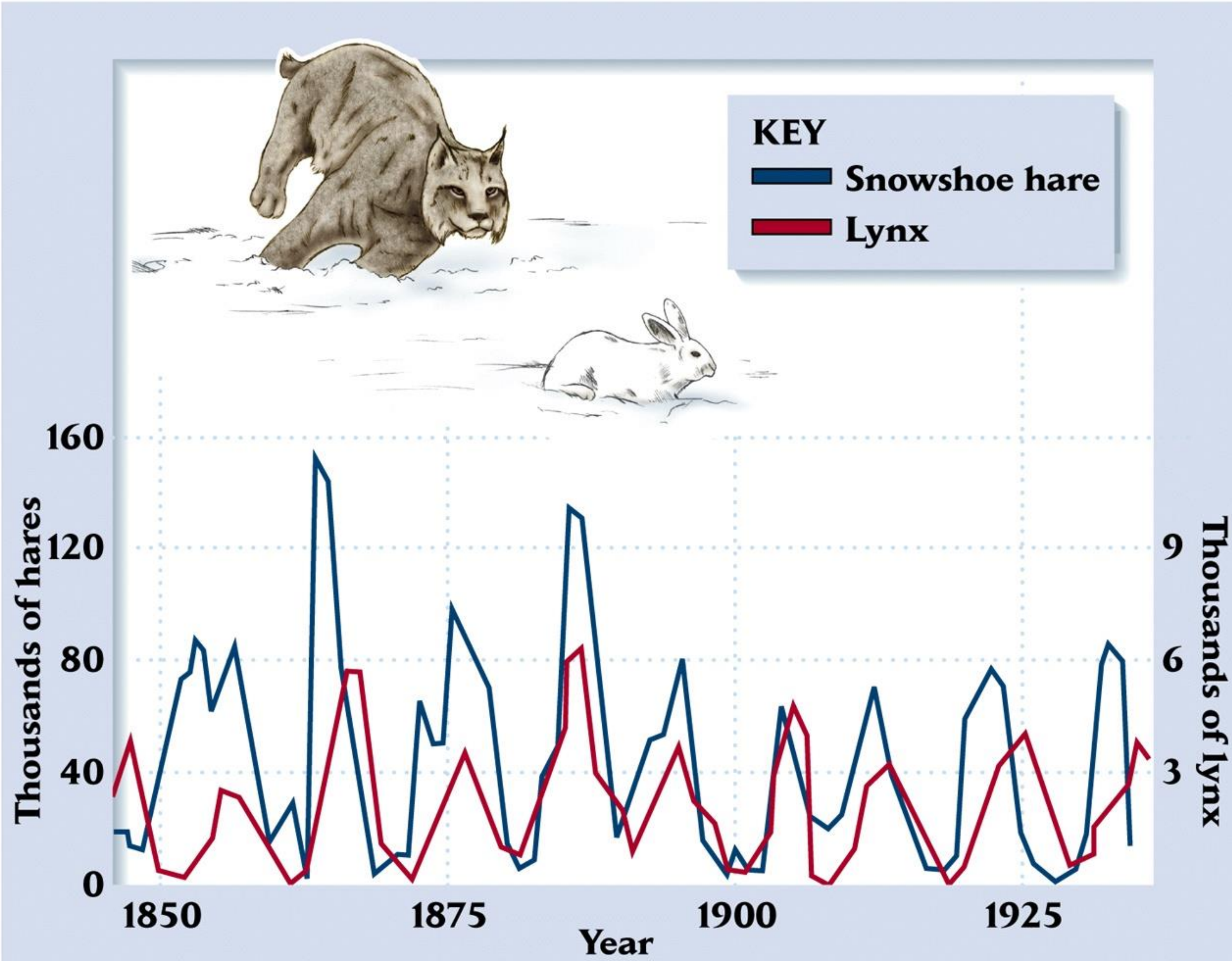
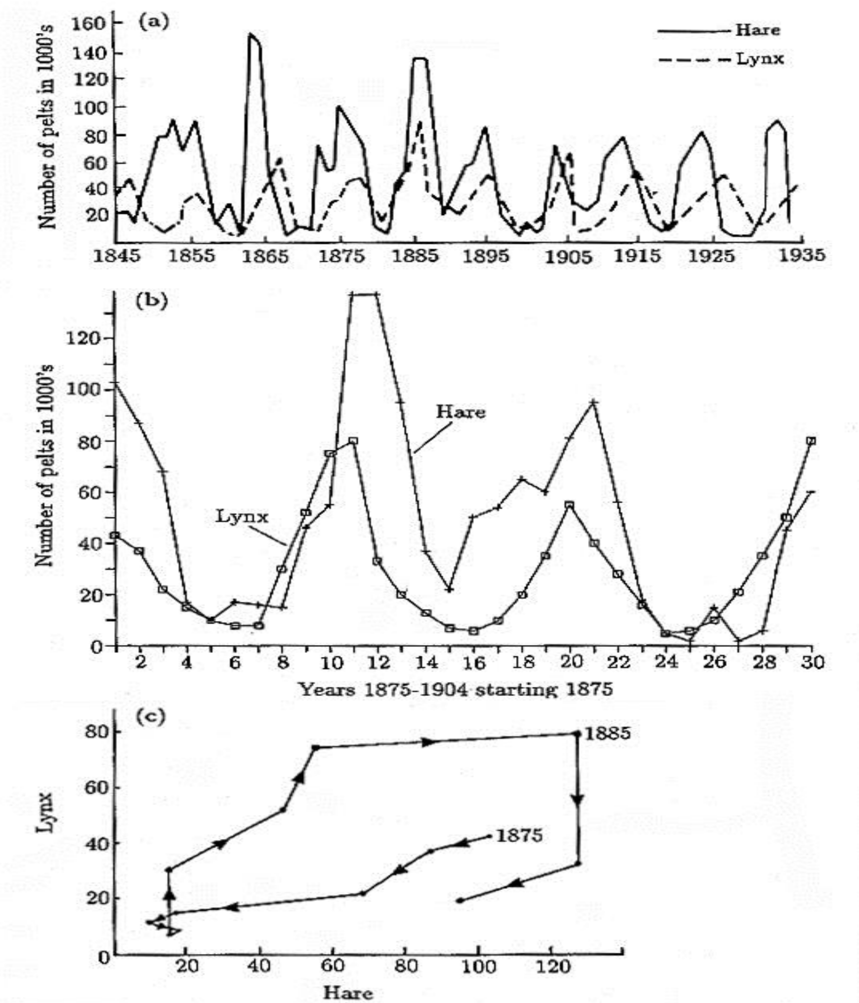
Modeling of population dynamics – predator-prey systems


KEY
Snowshoe hare
Lynx
160
120
80
40
0
Thousands of hares
9
6
3
Thousands of lynx
1850
1875
1900
1925
Year

Hare-Lynx data (Canada)



How to model dynamics of interacting populations?

- **A verbal model of predator-prey cycles:**

1. Predators eat prey and reduce their numbers
2. Predators go hungry and decline in number
3. With fewer predators, prey survive better and increase
4. Increasing prey populations allow predators to increase

- And repeat…

Lotka-Volterra Model

(1926)

$$\dot{H} = bH - sHP,$$
$$\dot{P} = -dP + esHP,$$

- H - prey
- P - predators
- b - prey growth rate : Malthus law
- d - predator mortality rate : natural mortality
- s, e - predation coefficients

Lotka-Volterra Model

Using

$$h = \frac{Hes}{d}, \quad p = \frac{Ps}{b}, \quad \tau = \sqrt{bd}\,t, \quad \rho = \sqrt{\frac{b}{d}},$$

L-V equations can be written as:

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = \rho h\,(1-p),$$
$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = -\frac{1}{\rho} p\,(1-h)$$

There are two equilibrium states of the above equations: (0,0) and (1,1).

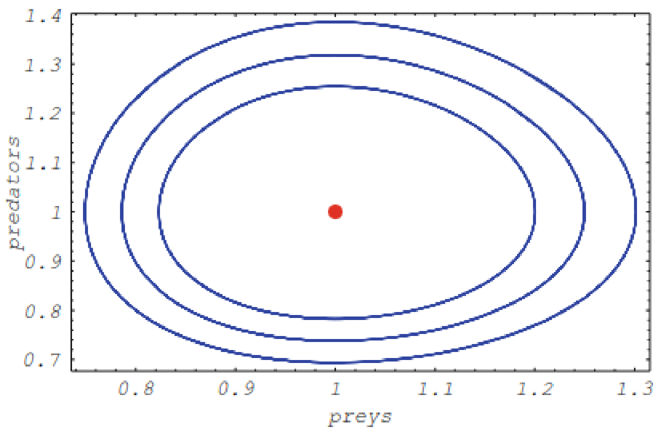
Calculating Jacobian one finds:

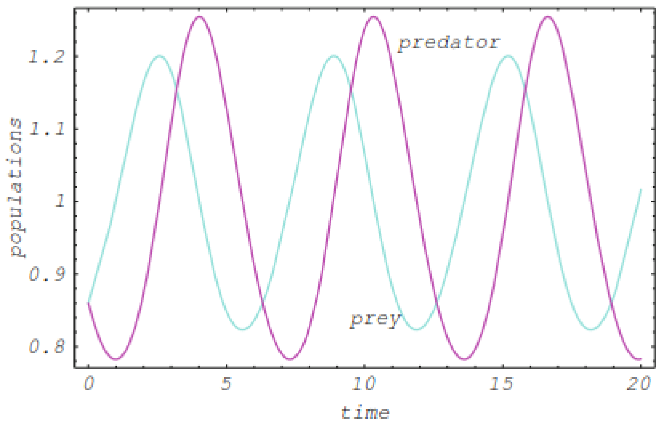
$$D\mathbf{f}(0,0) = \begin{bmatrix} \rho & 0 \\ 0 & -\frac{1}{\rho} \end{bmatrix}, \quad D\mathbf{f}(1,1) = \begin{bmatrix} 0 & -\rho \\ \frac{1}{\rho} & 0 \end{bmatrix}$$

Lotka-Volterra Model

One finds that equilibrium point:

- (0,0) is unstable
- (1,1) is neutral (both eigenvalues are imaginary)





Lotka–Volterra model. Scaled predator and prey populations as functions of scaled time

More realistic: L-V with carrying capacity

$$\dot{H} = bH\left(1 - \frac{H}{K}\right) - sHP,$$
$$\dot{P} = -dP + esHP,$$

K – carrying capacity

Find equilibrium points and examine their stability

We rewrite the above system as

$$\frac{dh}{d\tau} = \rho h\left(1 - \frac{h}{k} - p\right),$$
$$\frac{dp}{d\tau} = -\frac{1}{\rho}p\,(1-h).$$

$$k = \frac{Kes}{d}$$

The equilibrium points are $(0,0)$, $(k,0)$, and $(1, 1-1/k)$. $k > 1$

Calculating Jacobians we obtain:

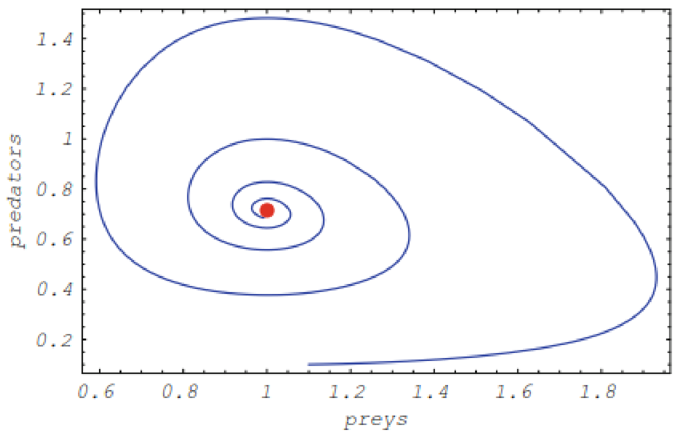
$$D\mathbf{f}(0,0)=\begin{bmatrix}\rho & 0\\ 0 & -1/\rho\end{bmatrix},\quad D\mathbf{f}(k,0)=\begin{bmatrix}-\rho & -\rho k\\ 0 & -(1-k)/\rho\end{bmatrix},$$

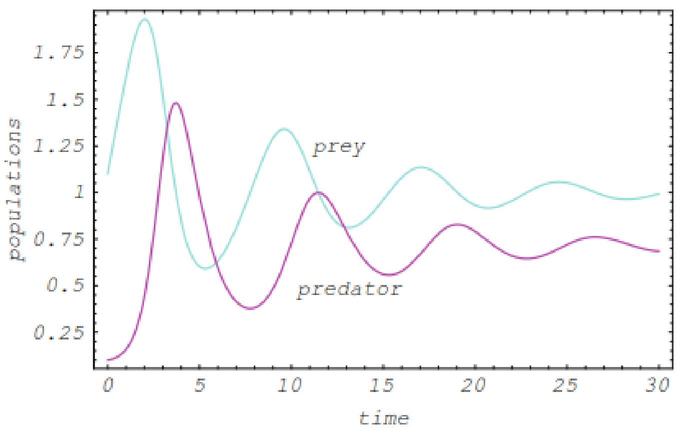
and

$$D\mathbf{f}\left(1,1-\frac{1}{k}\right)=\begin{bmatrix}-\rho/k & -\rho\\ (k-1)/\rho k & 0\end{bmatrix},$$

it follows that $(0,0)$ and $(k,0)$ are unstable, whereas $(1,1-1/k)$ is stable.

Carrying capacity (K) turns neutral equilibrium point into stable equilibrium point





L-V oscillations are structurally unstable. Small perturbation (large K) destroys oscillations.

Possible extensions:

Predator's functional response; that is, the relation between the predator's consumption rate and prey density.

The unbounded term –sHP should be replaced by bounded one (Holling 1959)

$$\dot{H} = r_H H \left(1 - \frac{H}{K}\right) - \frac{a_H P H}{b + H},$$
$$\dot{P} = \frac{a_P P H}{b + H} - cP,$$

Other effects:

Time delay:

$$\dot{N}(t) = \frac{\mathrm{d}N}{\mathrm{d}t} = rN(t)\left(1 - \frac{N(t-T)}{K}\right)$$ (Hutchinson model)

Behaviour: when rK is sufficiently large N(t) oscillates (single species model!)

Problem:

Consider the predator–prey model defined by the following system of two recurrence equations:

$$H_{n+1} = aH_n(1 - H_n) - bH_nP_n$$
$$P_{n+1} = dH_nP_n,$$

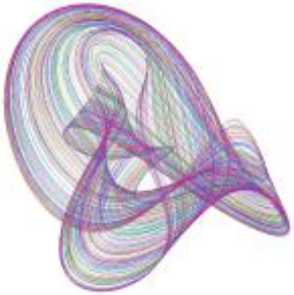
where H_n and P_n are the respective reduced prey and predator population densities, and a, b and d are positive constant. This model assumes that the predator can survive only in the presence of prey. Find the fixed points, and discuss their stability.

3-species model

- $X' = ax - bxy$ (prey-- mouse)
- $Y' = -cy + dxy - eyz$ (predator-- snake)
- $Z' = -fz + gxz$ (super-predator-- owl)
  - **a**: natural growth rate of prey in the absence of predation
  - **b, e,**: death rate due to predation
  - **c, f**: natural death rate of predator
  - **d, g**: growth rate due to predation

Poincaré-Bendixson

- No chaos in d=2 systems: trajectory approaches either a fixed point or a closed orbit
- In d>2 systems trajectories might settle on strange attractors



Limitations of Lotka-Volterra

- No time lag between predator and prey population responses

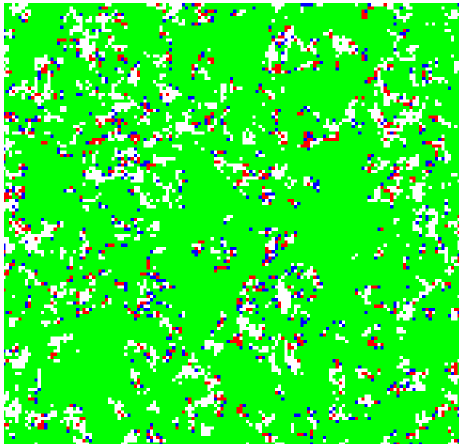
- Lack of heterogeneities in the distribution of species

$$\frac{\partial n}{\partial t} = rn(1 - n / K) + D\frac{\partial^2 n}{\partial x^2}$$

- Neutrally Stable= there is no attraction to some equilibrium point
  Any perturbation in the model will have it continue to cycle at a new amplitude until a new force acts on the model
  Eventually one of the populations oscillations may reach an axis signaling that a population has died out

- Based off of a Type I Functional response which is the least realistic type of functional response
  Type I response does not have any density dependence factors, therefore predator populations are linearly related to prey populations without any consideration of the density of the predator population

Prey – predators: lattice version



Rules:
- Choose a site
- Update if prey (breeds if possible)
- Update if predator (breeds,consumes prey if around, dies of hunger)
- Do nothing if empty

individual-oriented dynamics, spatial and temporal heterogeneities

Mean-field approximation:

$$\frac{dx(t)}{dt} = rx(t)[1-x(t)^{2d}] - (1-r)x(t)y(t),$$

$$\frac{dy(t)}{dt} = (1-r)x(t)y(t)[1-y(t)^{2d}] - (1-r)[1-x(t)]y(t).$$

Prey – predators: lattice version

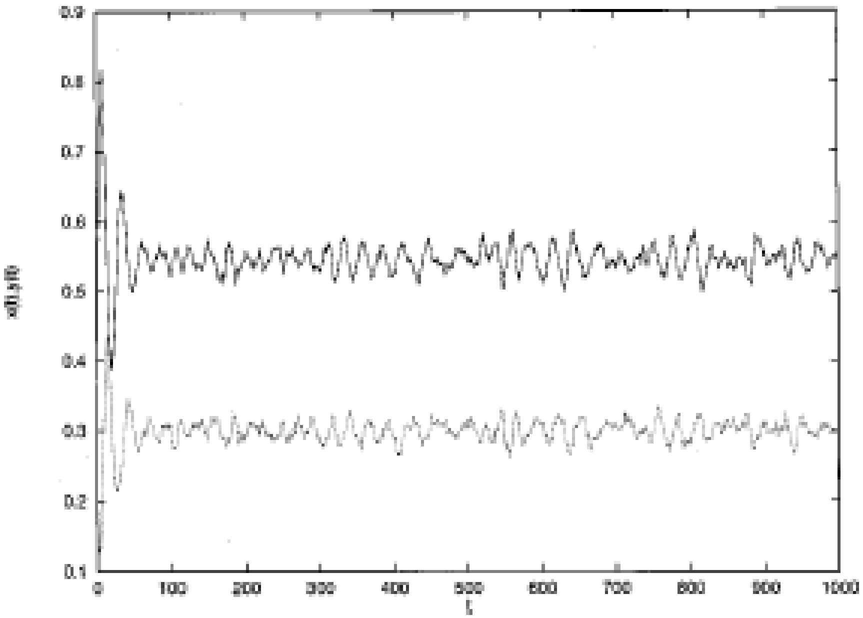


FIG. 4. Time evolution of $x(t)$ (solid line) and $y(t)$ (dotted line) for the two-dimensional model and $r=0.3$. Calculations were made for the linear size $L=200$.

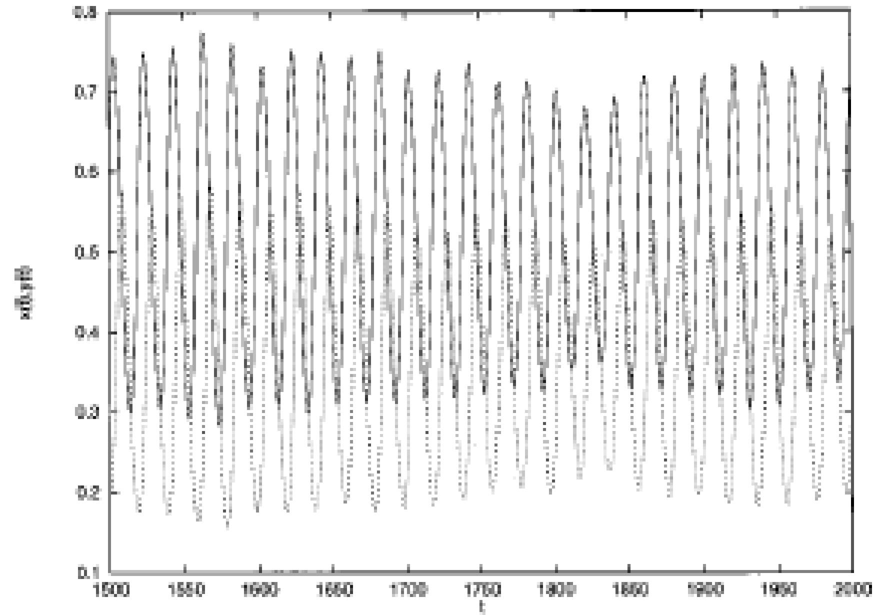


FIG. 6. Time evolution of $x(t)$ (solid line) and $y(t)$ (dotted line) for the three-dimensional model and $r=0.3$. Calculations were made for the linear size $L=30$.

Role of dimensionality? Why in d=2 systems oscillations vanish in the limit of large system size? In d=3 systems they do not vanish.

Spatiotemporal structures, Turing structures,…

More complex ecosystems: S-species Lotka-Volterra system

$$\frac{dN_i}{dt} = N_i\left(\varepsilon_i - \sum_{j=1}^{s} \alpha_{ij} N_j\right) \qquad i = 1,2,\ldots, s$$

N_i - population size of i-th species

-**May (1972):** Ecosystem becomes less stable when the number of species or their connectivities (interactions) increases

But real food webs are

- not random,
- perhaps not even at equilibrium

Multi-species ecosystems:

Simple models: Bak-Sneppen, Sole-Manrubia, Amaral-Meyer,...
Ignore population size, species and links added according to some simple rules.The main focus of these models is macroevolution.

Assembly models: Yodzis, Post-Pimm,...
Include population dynamics. One adds to ecosystem new species from a certain species pool until it becomes invasion resistant.

Evolutionary models: Caldarelli-Higgs-McKane, Stauffer et al.,
New species are created as modifications of existing species.

Lattice models with individual based dynamics: include population dynamics, but might also include evolution (mutations)